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SURVEY OF THE STATUS OF FINITE ELEMENT METHODS FOR PARTIAL DIFFERENITAL EQUATIONS

Roger Temam
Laboratoire d´Analyse Numerique
Universite´ Paris Sud, 91405 Orsay, France

ABSTRACT

The finite element methods (FEM) have proved to be a powerful technique for the solution of boundary value problems associated with partial differential equations of either elliptic, parabolic, or hyperbolic type. They also have a good potential for utilization on parallel computers particularly in relation to the concept of domain decomposition.

This report is intended as an introduction to the FEM for the nonspecialist. It contains a survey which is totally nonexhaustive, and it also contains as an illustration, a report on some new results concerning two specific applications, namely a free boundary fluid-structure interaction problem and the Euler equations for inviscid flows.

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INTRODUCTION

It is totally impossible to survey the theory of finite element methods within a few pages, and the object of this article is to describe for the nonspecialist some very basic ideas and concepts in finite elements approximations and to discuss some future trends in the theory without any attempt at being exhaustive. Beside this survey part, this article contains in Sections 8 and 9 a report on some new results concerning two specific problems, viz. a free boundary fluid structure interaction problem and the Euler equations for inviscid flows.

There is no agreement about the first appearance of the method. Finite element methods have probably been used for many years for computing and engineering purposes in a more or less explicit form. R. Courant mentions in [10] the approximation of a function in $\mathbb{R}^2$ by continuous piecewise linear functions on a triangulation, and this may be the first appearance in the mathematical literature.

Although it is difficult to track the first appearance of the method, there is no doubt that the first systematic and large scale utilizations of the finite element methods (FEM) occurred in the sixties in solid mechanics engineering. The period coincides of course with the first computers and the early stages of what we now call scientific computing. Probably the reasons which made FEM immediately popular among solid mechanics engineers is that, as we recall below, the foundations of the FEM coincide with some very fundamental concepts in solid mechanics. The method has then spread with different levels of response in fluid mechanics, optimization and control theory, and among mathematicians.

Alike the solid mechanists, mathematicians (numerical analysts and some more theoretically oriented mathematicians) have been working in FEM because the methods are appropriate for mathematical treatment and are very close in their fundamental concepts to the ideas and tools which are used in the mathematical treatment of the linear and nonlinear boundary value problems by functional analysis.

The mathematical and engineering literature on FEM for partial differential equations is abundant, and there is no way to survey it here. The questions that we address are the following ones: In Section 1, we recall the principle of weak formulations, and in Section 2, we recall the role of domain decomposition in the context of structural mechanics. We return to domain decomposition in Section 9 as it relates to future developments in the FEM in relation with parallel computation and some possible extensions of the method. Sections 3 to 5 are devoted to the description of very typical mathematical results. Section 3 describes the general mathematical framework and the most common finite elements. Section 4 provides some convergence and error results while Section 5 is an introduction to mixed and hybrid finite elements. Some specific applications (among many others) of the FEM are then described. Section 6 is related to the Navier-Stokes equations. Section 7 deals with fluid-structure interactions problems, and Section 8 deals with the applications of FEM to the solution of the Euler equations. Finally, as indicated above, we return in Section 9 to domain decomposition and the role that this can play in future developments for FEM.

FOUNDATIONS OF THE FEM

The finite element methods lie on two fundamental ideas:

- the weak formulation of a boundary value problem,
- the domain decomposition, i.e., the decomposition of the domain corresponding to the problem into smaller subdomains, the elements.

As mentioned above, both ideas are closely related to basic concepts of solid mechanics. The weak formulation of a boundary value problem coincides with the virtual work theorems and energy principles in the statics of solids. Domain decomposition is also an extrapolation of the natural approach in structural mechanics where large structures consist of smaller substructures which are properly connected or assembled, and the study of the large structure is reduced to that of the elementary structures and their connections.

1. Weak Formulations

We begin by recalling briefly the weak formulation of some boundary value problems in solid and fluid mechanics. Other examples of weak formulations will appear below (abstract boundary value problems).

1.a. Weak Formulations in Solid Mechanics

Consider a solid body which fills at rest a region Ω of $\mathbb{R}^3$ with boundary Γ. We assume that the body is subjected to volumic forces of density $f = (f_1,f_2,f_3)$ in Ω and to surface (traction) forces of surface density $F = (F_1,F_2,F_3)$ on some part Γ_1 of Γ and reaches a new equilibrium position. The unknowns of the problem are:

- the field displacements, $u = (u_1,u_2,u_3)$, $u(x)$ $x\in\Omega$, representing the

displacement between the position at rest of a particle $x \in \Omega$ and its new equilibrium position $x + u(x)$.

- the boundary stress tensor field, $\sigma = (\sigma_{ij})$.

Under the assumption of small displacements, the equilibrium equations read

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0 \qquad \text{in } \Omega \tag{1.1}$$

$$\sum_{j=1}^{3} \sigma_{ij} \nu_j = F_i \qquad \text{on } \Gamma_1, \tag{1.2}$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ is the unit outward normal on Γ.

Usually the displacement u is given on the complementary part Γ_0 of Γ_1, $\Gamma_0 = \Gamma \setminus \Gamma_1$

$$u = U \quad \text{on } \Gamma_0. \tag{1.3}$$

The so-called set of statically admissible stress tensors $S_{ad}(f,F)$ is the set of tensor fields σ satisfying (1.1) and (1.2). The set $C_{ad}(U)$ is the set of kinematically admissible displacement fields, i.e., the set of u's satisfying (1.3).

The equations (1.1) - (1.3) which hold for any material are supplemented by the constitutive equations of the material which depend on the material and connect stresses and displacements. Without describing these relations, we can already see the weak formulation of the problem. Let σ, u be solution of (1.1) - (1.3) and let v be another kinematically admissible field of displacements, $v \in C_{ad}(U)$ (and $w = v - u \in C_{ad}^0 \equiv C_{ad}(0)$). We multiply (1.1) by

w_i, add these relations for $i = 1,2,3$, integrate over Ω, and use Green formula and (1.2) (1.3). We obtain

$$(1.4) \qquad \int_\Omega \sigma_{ij}\, \varepsilon_{ij}(w)dx = \int_\Omega f_i w_i dx + \int_{\Gamma_i} F_i w_i d\Gamma, \qquad \text{for all } w \in C^0_{ad},$$

where the Einstein summation convention has been used and $\varepsilon(w) = (\varepsilon_{ij}(w))$ is the strain tensor

$$\varepsilon_{ij}(w) = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right).$$

If we remember that $\sigma = \sigma(u)$ due to the constitutive law, we find that (1.4) is the weak formulation for the displacements[1]. For instance, in the simplest case of linear elasticity we have pointwise

$$(1.5) \qquad \sigma_{ij}(x) = A_{ijkl}\varepsilon_{kl}(u)(x), \qquad \text{for all } x \in \Omega,$$

where the coefficients A_{ijkl} define a linear positive invertible operator A in the space of symmetric tensors of order two. Whence (1.4) becomes

$$(1.6) \int_\Omega A_{ijkl}(u)\varepsilon_{kl}(u)\varepsilon_{ij}(w)dx = \int_\Omega f_i w_i dx + \int_{\Gamma_1} f_i w_i d\Gamma, \qquad \text{for all } w \in C^0_{ad}.$$

In linear and nonlinear elasticity, the weak formulation (1.4) (or (1.6)) coincides with the relation given by the virtual work theorem. It leads also to energy principles.

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[1] A similar formulation is available for the stresses σ.

1.b. Weak Formulations in Fluid Mechanics

Weak formulations in fluid mechanics do not have a physical interpretation as natural as in solid mechanics. They have been introduced by J. Leray ([16], [17], [18]) for the study of weak (i.e., nonregular) solutions of the Navier-Stokes equations in an attempt to explain turbulence by the appearance of singularities in the curl vector of the flow. Although we do not know yet if such singularities arise in space dimension three, there is no doubt that the contribution of J. Leray has been a fundamental step for the mathematical treatment of the Navier-Stokes equations by the modern methods of functional analysis and also for the numerical treatment of the equations in Computational Fluid Dynamics.

Consider for example the Navier-Stokes equations of an incompressible fluid in the stationary case. The fluid fills a bounded region Ω of $\mathbb{R}^3$ with boundary Γ. In the Eulerian representation of the flow, the unknowns are the velocity field $u = (u_1, u_2, u_3)$ and the pressure field p; $u = u(x)$, $x \in \Omega$ is the velocity of the particle of fluid at x, and $p(x)$ is the pressure at point x. We have the equations

$$-\nu \Delta u + (u \cdot \nabla) u + \nabla p = f \quad \text{in } \Omega \tag{1.7}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega, \tag{1.8}$$

where $\nu > 0$ is the kinematic viscosity and f represents volumic forces. Equation (1.7) is the equation of conservation of momentum. Equation (1.8) is the incompressibility equation, i.e., the equation of mass conservation. If Γ is materialized and moving with velocity U, then the nonslip

condition on Γ is

(1.9) $$u = U \quad \text{on} \quad \Gamma.$$

Let $V(U)$ be the space of functions satisfying (1.8) and (1.9). Then $u \in V(U)$ and if v is a test function in $V(U)$, $w = v - u \in V(0)$. We take the scalar product of (1.7) with w (pointwise in $\mathbb{R}^3$), integrate over Ω, and use Green's formula. We have

$$-\int_\Omega \Delta u \, w dx = -\int_\Omega \Delta u_i w_i dx = \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} dx$$

$$\int_\Omega \text{grad } p \cdot w dx = \int_\Gamma pw \cdot \nu d\Gamma - \int_\Omega p \, \text{div } w dx = 0.$$

Whence p disappears and we obtain the weak formulation of (1.7) - (1.9):

(1.10)
$$\begin{cases} u \in V(U) \quad \text{and for every} \quad w \in V(0) \\ \nu \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} dx + \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial u_j}{\partial x_i} w_j dx = \\ = \sum_{i=1}^{3} \int f_i w_i dx. \end{cases}$$

It is equivalent to say that u satisfies (1.10) or that u satisfies (1.7) - (1.9). The striking fact in formulation (1.10) is that <u>the pressure disappears</u> and we are left with an equation involving u only. Once u is found we know from mathematical results that there exists p which is defined up to an additive constant by (1.7). However, in the practice of numerical

computations p is obtained differently, in general, as the Lagrange multiplier of the constraint $\operatorname{div} u = 0$ (see Section 5 below).

2. Domain Decomposition

In structural mechanics it is natural to compute a complicated structure by considering the smaller substructures of which it is made. Each substructure is well modeled, its behavior is well understood, and then the mechanical engineers model the interaction (contact laws, etc.) of the different components to obtain the description of the full structure.

As mentioned before, finite elements in solid mechanics have started as an extrapolation of this idea to continuous bodies: the full solid body is decomposed into smaller elements; a simplified constitutive law is adopted on each element; and a simplified version of the constitutive law leads to simplified interactions laws between the contiguous elements.

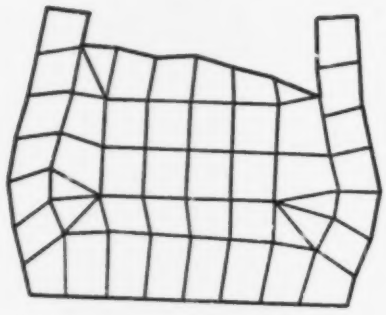


Figure 2.1

Similarly, the particle and cells methods in fluid mechanics which are very close to the finite element methods are based on a simplified analysis of the flow in small cells with simplified fluid transfer laws. The generalization and mathematization of the FEM have led to a more systematic view and a more systematic approach.

Beside discretization, there are several other good reasons to decompose a large domain into smaller subdomains. These reasons are also at the heart of future developments in scientific computation and probably finite elements. We will return on this important question in Section 8.

MAIN METHODS - MAIN MATHEMATICAL RESULTS

We give an overview of some typical finite element methods and some typical mathematical results which have been obtained.

3. The Usual Finite Elements Methods

3.a. A Model Problem

We consider as a model problem the following mathematical problem.

We denote by Ω an open bounded domain of $\mathbb{R}^n$, with boundary Γ, and we consider a Laplace equation,

$$-\Delta u + u = f \quad \text{in} \quad \Omega, \tag{3.1}$$

with associated boundary conditions of Dirichlet and Neuman type

(3.2) $$u = 0 \quad \text{on} \quad \Gamma_0$$

(3.3) $$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1$$

where Γ_0, Γ_1 is a partition of Γ. In the two-limit cases $\Gamma_0 = \Gamma, \Gamma_1 = \emptyset$ and $\Gamma_0 = \emptyset, \Gamma_1 = \Gamma$ we obtain respectively the Dirichlet and Neuman problems; the general case is a mixed boundary value problem.

Let V be the space of functions u satisfying (3.2) and possessing a certain level of regularity which we do not specify at the moment. The solution u of (3.1) - (3.3) belongs to V and if v is a test function in V, we multiply (3.1) by v, integrate over Ω, and apply Green's formula. Thanks to (3.3) (and $v = 0$ on Γ_0) we find

(3.4) $$-\int_\Omega \Delta u \, v dx = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx$$

and thus

(3.5) $$\begin{cases} u \in V \quad \text{and} \\ a(u,v) = (f,v), \qquad \text{for all} \quad v \in V \end{cases}$$

where

(3.6) $$a(u,v) = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx + \int_\Omega uv dx$$

and

(3.7) $$(f,v) = \int_{\Omega} f(x)\, v(x)dx$$

is the scalar product in $L^2(\Omega)$.

Conversely, it can be proved (under suitable regularity assumptions) that if u satisfies (3.5) then u is the solution of (3.1) - (3.3). Equation (3.1) is derived from (3.5) by appropriate methods using distribution derivatives; (3.2) follows from "$u \in V$" whereas (3.3) is a boundary condition hidden in (3.5). This is a general fact with weak formulations like (3.5): some boundary conditions of the problem are contained in the definition of the space V, and some boundary conditions are contained in the equation (3.5).

Let us give a more precise definition of the space V. Roughly speaking, the space V will be the space of all functions u vanishing on Γ_0 and such that $a(u,u) < \infty$. More precisely it is easy to see that the expression

$$\{a(u,u)\}^{1/2}$$

is a norm on the space of continuously differentiable functions on $\overline{\Omega}$ which vanish on Γ_0. We define V as the completion of this space for this norm; we obtain the space

(3.8) $$V = \{v \in H^1(\Omega),\ v|_{\Gamma_0} = 0\}$$

where $H^1(\Omega)$ is the Sobolev space

(3.9) $$H^1(\Omega) = \{v \in L^2(\Omega),\ \frac{\partial v}{\partial x_i} \in L^2(\Omega), \quad i=1,\ldots,n\}.$$

More generally $H^m(\Omega)$, the Sobolev space of order m, is the space of functions u square integrable in Ω ($u \in L^2(\Omega)$) such that all derivatives of order $\leq m$ are square integrable also.

3.b. Abstract Boundary Value Problem

The situation in (3.5) is typical of many linear elliptic boundary value problems. The abstract setting is the following one:

- We are given a Hilbert space V (norm $\|\cdot\|_V$) and a bilinear form a on $V \times V$ which is continuous, i.e.,

(3.10) There exists $M < \infty$ such that
$a(u,v) \leq M \|u\|_V \|v\|_V$, for all $u,v \in V$

and coercive, i.e.,

(3.11) There exists $\alpha > 0$ such that
$a(u,u) \geq \alpha \|u\|_V 2$, for all $u \in V$.

- We are given also a linear continuous form ℓ on V, i.e., an element of the dual V' of V

and then the problem is

(3.12) $\begin{cases} \text{To find } u \in V \text{ such that} \\ a(u,v) = \langle \ell, v \rangle, \text{ for all } v \in V. \end{cases}$

Despite its simplicity, (3.12) is applicable to many interesting boundary value problems in mechanics and physics. The existence and uniqueness of a solution u of (3.12) is classically provided by the Lax-Milgram Theorem (see for instance [28]).

More generally nonlinear elliptic boundary value problems can be set in a form similar to (3.12) if we allow V to be a Banach space and a to be nonlinear with respect to its first argument (i.e., a maps $V \times V$ into $\mathbb{R}$ and is linear with respect to its second argument). For instance, it follows readily from (1.10) that the stationary Navier-Stokes equations (1.7) - (1.9) with $U = 0$ can be written in this form. Similarly, consider the problem (3.1) - (3.3) and replace the linear equation

$$-\Delta u + u = f \quad \text{in} \quad \Omega$$

by the nonlinear one

$$-\Delta u + p(u) = f \quad \text{in} \quad \Omega, \tag{3.13}$$

where p is a polynomial of odd degree with a positive leading coefficient. Then (3.13), (3.2), and (3.3) can be set in a form similar to (3.12)

$$V = \{v \in H^1(\Omega) \cap L^{\alpha+1}(\Omega),\ v|_{\Gamma_0} = 0\},$$

$$a(u,v) = \sum_{i=1}^{n} \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx + \int_{\Omega} p(u)v\, dx,$$

where α is the degree of the polynomial p (see [28]).

In the nonlinear case, there are no general assumptions on a covering all the interesting situations, and we will restrict ourselves to specific examples.

3.c. General Form of Finite Element Approximations

The discretization of the abstract boundary value problem (3.12) consists in choosing

- a family $(V_h)_{h\in H}$ of finite dimensional approximations of V
- a family $(a_u(u_h, v_h))_{h\in H}$ of bilinear forms on $V_h \times V_h$ which approximate a.

Roughly speaking there are two types of discretizations produced by the finite elements:

- the conforming finite elements in which the V_h are subspaces of V of higher and higher dimensions as the parameter $h \to 0$,
- the nonconforming finite elements in which the V_h are not subspaces of V.

Of course finite elements have been only used in space dimension $n = 2$ and at a less developed stage when $n = 3$. We consider first the case where Ω is a polygonal set. A basic ingredient of FEM is a triangulation of Ω. By this, we mean a suitable covering of Ω by either

- a family of triangles,
- a family of rectangles whose sides are parallel to the axes (or more general quadrilateral sets),
- or a combination of triangles and rectangles (or quadrilateral sets).

The triangles or rectangles are the (finite) "elements." The space V_h consists of functions of a given type (usually a polynomial) on each element

which are properly connected. The values of the functions of V_h or their derivatives at some particular points of the elements (vertices, mid-edges,...) are the nodal values which fully determine the functions in V_h. A natural basis of V_h consists of the so-called shape functions: These are the functions of V_h whose nodal values are 1 for one of them and 0 for all the others. In most cases these functions have a "small" support, and this leads to fairly sparse matrices for the discretized problem. When a function v is defined on Ω (or on an element K), its interpolant on Ω (or K) denoted $r_h v$ (or $r_K v$) is the function of V_h (or the elementary function on K) which has the same nodal values as v.

3.d. Conforming Finite Elements (n = 2).

For second order elliptic boundary value problems, the basic space V is $H^1(\Omega)$ or a product of such spaces or a subspace of such spaces.

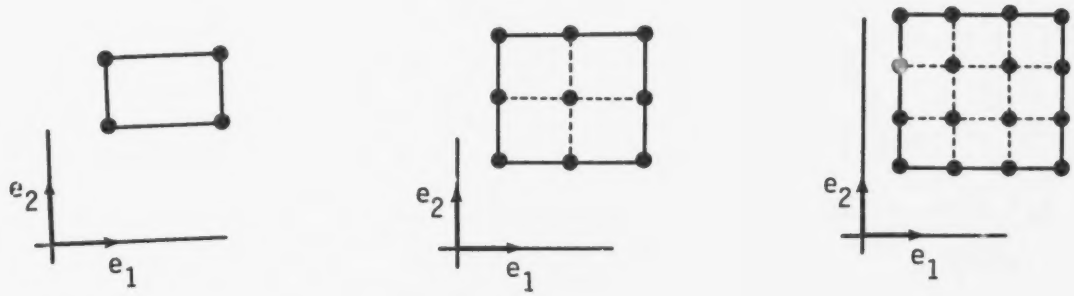
The simplest and most common elements used in this case are the P_1 elements on triangles and the Q_1 elements on rectangles. P_1 (respectively P_n) is the set of polynomials of degree ≤ 1 (resp. $\leq m$), whereas Q_1 (resp. Q_m) is the set of polynomials of degree ≤ 1 (resp. $\leq m$) with respect to each variable.

Some other typical elements used for second order boundary value problems are depicted in Figure 3.1. We will return to the P_1 and Q_1 elements after we briefly describe the elements in Figure 3.1.

Triangles

linear: polynomials of degree ≤ 1 on the triangles, nodal values = values at vertices.

Linear, quadratic, cubic triangles



Linear, quadratic, cubic rectangles

Conforming Hermite elements

Reduced Cubic Triangle

Reduced quadratic Triangle

Figure 3.1: Conforming Finite Elements (n = 2)

quadratic: polynomials of degree ≤ 2 on the triangles; nodal values = values at vertices and midedges.

cubic: polynomials of degree ≤ 3 on triangles; nodal values = values at vertices, barycenter and 1/3 points on edges.

reduced cubic: polynomials of degree ≤ 3, vanishing at the barycenter on each triangle; nodal values = values at vertices and 1/3 points on edges.

Rectangles

linear: polynomials of degree ≤ 1 in each variable on rectangles; nodal values = values at vertices.

quadratic: polynomials of degree ≤ 2 in each variable; nodal values = values at vertices, midedges, and center.

cubic: polynomials of degree ≤ 3 in each variable; nodal values = values of function at 16 different points (see Figure 3.1).

reduced quadratic: polynomials of degree ≤ 2 in each variable satisfying a linear relation (on each rectangle); nodal values = values of function at vertices and midedges.

All functions obtained by these elements are globally C^0 (continuous) except the quadratic Hermite triangle which produces C^1 approximants (continuously differentiable functions).

More special elements can be found in the literature; cf. for instance the book of P. G. Ciarlet [9] on the mathematical side, and the book of Zienkiewicz [33], the work of Argyris [1] and others on the engineering or mechanical sides.

The more sophisticated elements produce better (more precise) results but need more computing time and a good expertise in finite elements technology. In a nonspecialized industrial environment, the tendency seems to be the utilization of simple elements of degree one or at most two with a suitable refinement of the mesh.

As mentioned above the simplest and most commonly used elements are the P_1 elements on triangles and the Q_1 elements on rectangles with sides parallel to the x and y axes. Let us mention also the quadrilateral elements described hereafter.

Let $\hat{K}$ denote the square $(0,1) \times (0,1)$ in the ξ, η plane. We observe that a mapping F with Q_1 components

$$F(\xi,\eta) = \begin{cases} a + b\xi + c\eta + d\xi\eta \\ a' + b'\xi + c'\eta + d'\xi\eta \end{cases}$$

can map $\hat{K}$ on any arbitrary quadrilateral K of the x,y plane. The image by F of a line in the ξ,η plane is generally a curved line of the x,y plane. However the lines x = constant, y = constant, and in particular the boundary of $\hat{K}$ are mapped by F onto straight lines of the x,y plane. A natural element on the quadrilateral K is the image by F^{-1} of

the Q_1

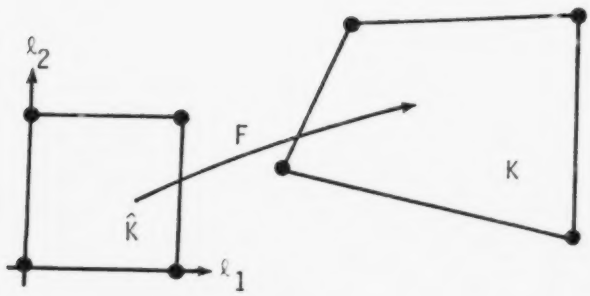


Figure 3.2

element on $\hat{K}$:

$$q = q(\xi,\eta)\in Q_1 \rightarrow qoF^{-1} = qoF^{-1}(x,y).$$

In general these elements <u>are not polynomials on</u> K. They are, however, easy to use and their explicit expression is rarely used.

3.e. Conforming Finite Elements (n = 3)

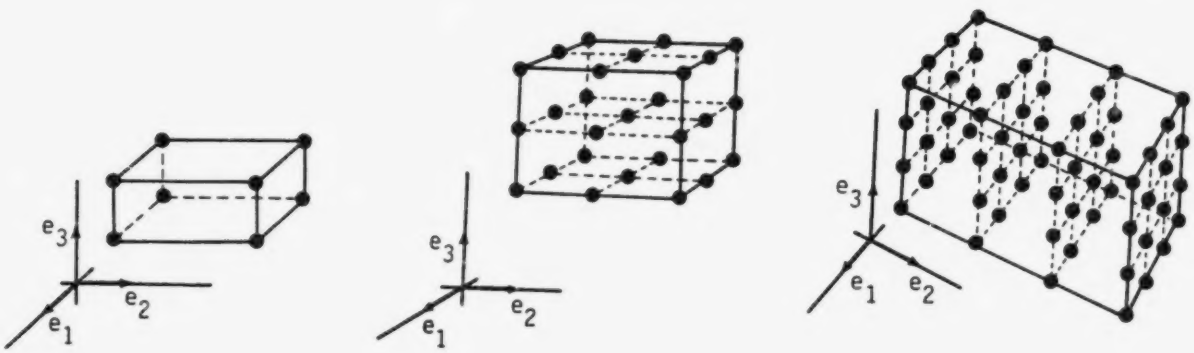
The triangulation is now the covering of Ω (= a polygonal set) by either tetrahedrons or 3-D rectangles whose edges are parallel to the axes or combinations of those.

The most common elements are the

- linear, quadratic, and cubic tetrahedrons.
- linear, quadratic, and cubic 3-D rectangles.

The definitions of these elements are the same as above in the two-dimensional case replacing triangle by tetrahedron and rectangle by 3-D rectangle. For

Linear, quadratic, cubic tetrahedrons



Linear, quadratic, cubic rectangles

Figure 3.3: Conforming Finite Elements (n = 3)

the cubic (tetrahedron and 3-D rectangle) elements, the nodal values are shown on Figure 3.3. All these elements lead to functions which are globally C^0 (continuous) but not more.

3.f. Nonconforming Finite Elements

As indicated above, nonconforming finite elements produce approximate function spaces V_h which are not subspaces of V. For instance, the linear nonconforming triangle described below produces, when applied to Problem (3.12), approximate functions which are highly discontinuous. Still, it may be useful to use such elements in at least two cases:

- Fluid flow problems where, due to the incompressibility condition $\mathrm{div}\, u = 0$, the linear triangle elements cannot be used in a straightforward manner.
- Higher order problems, like the biharmonic problem where most elements described above fail to produce C^1 functions, and thus the approximate spaces V_h would not be included in $H^2(\Omega)$ (= the natural space for a biharmonic problem).

Nonconforming linear elements

2-D Case (Triangles)

Polynomials of degree ≤ 1 on the triangles

Nodal values = values at midedges

3-D Case (Tetrahedron)

Polynoms of degree ≤ 1 on the tetrahedrons

Nodal values = values at barycenter of faces

The global functions are totally discontinuous with discontinuities along the edges of triangles (or faces of tetrahedrons) except for the barycenters (of edges or faces). The method is nevertheless convergent and efficient, particularly for fluid flows: see the book of F. Thomasset [32] which is fully devoted to the utilization of these elements in 2-D flows.

3.g. Curved boundaries

Curved boundaries can be approximated by polygonal lines. Alternatively one can use the so-called isoparametric elements: the element is the image by an appropriate (simple) mapping of a triangle or a rectangle and the function reduces on the element to the composition of a polynomial with that mapping. A similar situation occurred with the Q_1 quadrilateral elements.

4. Convergence and Error Estimate

Concerning convergence and error estimates the situation is different for linear and nonlinear problems.

4.a. Linear Problems

Two type of results have been derived in relation with error computation and convergence (see for instance P. G. Ciarlet [9]):

- interpolation error,
- approximation error.

When V_h is a conforming finite element space and u is a function in V (or usually in a smaller space), we consider the interpolant $r_h u$ of u in V_h (this is the finite element function which assumes the same nodal values as u, whereas $r_K u$ is the interpolant of u on an element K); the interpolation results give an upper bound of the norm of $u - r_h u$ in V and other spaces. The approximation results are of a different nature: when $u \in V$ is a solution of a problem like (3.5) and $u_h \in V_h$ is a solution of the associated discrete problem, then the error between u and u_h is estimated for various norms. In the optimal cases the error between u and u_h is of the same order as that of the distance of u to V_h.

The general results are too abstract to be presented in detail here; we will just recapitulate the error estimates corresponding to the elements described above.

For an element K let ρ_K denote the radius of the smallest ball containing K, let ρ'_K denote the radius of the largest ball included in K, and let $\sigma_K = \rho_K / \rho'_K$.

The analysis is made under the assumptions that

(4.1)
$$\rho_h = \sup_{K \in T_h} \rho_K \to 0$$

and

(4.2) $\sigma_h = \sup_{K \in T_h} \sigma_K$ remains bounded from above.

If v is a function in V and $r_h v$ its interpolated function in V_h, we consider the H^m semi-norm of $v - r_h v$ on an element K of the triangu-

lation T_h and on the whole domain Ω:

$$|v - r_h v|_{m,K} = \left\{ \sum_{[\alpha]=m} \int_K |D^\alpha (v - r_h v)|^2 dx \right\}^{1/2}$$

$$|v - r_h v|_{m,\Omega} = \left\{ \sum_{[\alpha]=m} \int_\Omega |D^\alpha (v - r_h v)|^2 dx \right\}^{1/2},$$

where D^α is a partial derivative of order $[\alpha] = m$ and the sum is extended to all such derivatives.

For the elements described above, the interpolation result is the following one:

On an element K assume that the interpolation operator r_K is such that $r_K p = p$ for each polynom p of degree $\leq k$, and assume that r_K is linear continuous from $H^{k+1}(K)$ into $H^m(K)$, $0 \leq m \leq k+1$. Then

$$|v - r_K v|_{m,K} \leq c \frac{\rho_K^{k+1}}{(\rho_K^-)^m} |v|_{k+1,\,K} \tag{4.3}$$

for all $v \in H^{k+1}(K)$.

We can also assemble the results on the different elements K of a triangulation T_h and obtain a similar bound on all of Ω (when Ω is a polygon fully covered by the elements):

$$|v - r_h v|_{m,\Omega} \leq C \rho_h^{k+1-m} \sigma_h^m |v|_{k+1,\Omega}, \tag{4.4}$$

for all $v \in H^{k+1}(\Omega)$.

Finally in dimensions 2 or 3

- for the linear elements (triangles, rectangles, tetrahedrons, 3-D rectangles) if:

$$k = 1,\ v \in H^2(\Omega),\ \text{then}\ \ |v - r_h v|_{m,\Omega} = O(\rho_h^{2-m}),$$

$$0 \leq m \leq 2$$

- for the quadratic elements (triangles, rectangles, tetrahedrons, 3-D rectangles) if:

$$k = 2,\ v \in H^3(\Omega)\ \ \text{then},\ |v - r_h v|_{m,\Omega} = O(\rho_h^{3-m}),$$

$$0 \leq m \leq 3$$

- for the cubic elements (triangles, rectangles, tetrahedrons, 3-D rectangles) if:

$$k = 3,\ v \in H^4(\Omega),\ \text{then}\ |v - r_h v|_{m,\Omega} = O(\rho_h^{4-m})$$

$$0 \leq m \leq 4.$$

Concerning the approximation error, they are optimal (i.e., the approximation error is of the order of the best interpolation error), for instance, with the elements above, for Problem (3.1) - (3.5) when $\Gamma_0 = \Gamma$, $\Gamma_1 = \emptyset$ (Dirichlet problem) or $\Gamma_0 = \emptyset$, $\Gamma_1 = \Gamma$ (Neuman problem), and Ω is a polygon fully covered by the elements of the triangulation T_h.

4.b. Nonlinear Problems

For nonlinear problems the situation is more difficult and the results are less complete. Usually convergence results can be proved by using energy type inequalities and convergence techniques which are appropriate for the type of equations considered: see for instance [28] for the nonlinear problem (3.13), (3.2), (3.3), and [29] for the Navier-Stokes equations. When compactness methods are used some involved compactness arguments for finite elements may be necessary: cf. in R. Temam [29] the proof of convergence of the nonconforming P_1 finite element methods for the Navier Stokes equations. Also by lack of uniqueness for nonlinear elliptic problems the convergence may be limited to a subsequence or may assume as usual that we are "close" to the solution.

Error estimates are also more difficult to obtain than in the linear case. They assume usually more regularity on the equation and/or the solution that is necessary for convergence.

5. Mixed and Hybrid Finite Elements

5.a. Minimax Formulation of a Boundary Value Problem

Consider an abstract boundary value problem of the form (3.12)

$$\text{(5.1)} \qquad \begin{cases} \text{To find } u \in V \text{ such that} \\ a(u,v) = \langle \ell, v \rangle, \text{ for all } u \in V. \end{cases}$$

When the bilinear form a is futhermore symmetric, then (5.1) is equivalent to a convex minimization problem:

$$(5.2) \qquad \begin{cases} \text{To minimize for } v \in V, \\ J(v) = \frac{1}{2}\, a(v,v) - \langle \ell, v \rangle. \end{cases}$$

The infimum of J on V is attained at a unique point of V which is called a solution (or a minimizer) for the variational problem (5.2). In fact the solution of (5.1) is the same as that of (5.2).

The mixed finite elements are closely related to duality. A natural framework for both questions arises when V is a linear subspace of a Hilbert space X of the form

$$(5.3) \qquad V = \{ v \in X,\ b(v,\phi) = 0 \quad \text{for all} \quad \phi \in Y \},$$

where Y is another Hilbert space and b is a bilinear continuous form on $X \times Y$. We assume furthermore that a is extended as a bilinear continuous form on X and that ℓ is extended as a linear continuous form on X.

In this case we introduce the Lagrangian of the problem (cf. Ekeland-Temam [11]):

$$(5.3) \qquad L(v,\psi) = J(v) + b(v,\Psi).$$

It is easily verified that

$$\sup_{\phi \in Y} L(v,\phi) = \begin{cases} J(v) & \text{if } v \in V \\ +\infty & \text{if } v \in X \setminus V \end{cases}$$

and that the minimization problem (for $v \in V$):

$$\inf_{v \in V} \{\sup_{\psi \in Y} L(v,\psi)\} \tag{5.4}$$

has the same solution and the same infimum as (5.2).

Now we can associate with (5.3) the so-called <u>dual problem</u> of (5.4) which is a maximization problem in Y

$$\sup_{\psi \in Y} \{\inf_{v \in V} L(v,\psi)\} \tag{5.5}$$

It is shown in [11] that if L (i.e., here b) satisfies a suitable condition, then (5.5) has a unique solution denoted ϕ. Furthermore, the pair $\{u,\phi\} \in X \times Y$ is a solution of (5.5) and (5.4) (or (5.1)) if and only if $\frac{\partial L}{\partial v}(u,\phi) = \frac{\partial L}{\partial \psi}(u,\phi) = 0$, i.e.,

$$\begin{cases} a(u,v) + b(v,\phi) = \langle \ell, v \rangle, & \text{for all } v \in X \\ b(u,\psi) = 0, & \text{for all } \psi \in Y \end{cases} \tag{5.6}$$

The initial problem (5.1) (5.2) is written in X as a constrained minimization problem

$$\begin{cases} \text{To minimize } J(v) \text{ for } v \in X, \text{ subject to the constraint} \\ b(v,\psi) = 0 \quad \text{for all } \psi \in Y. \end{cases} \tag{5.7}$$

The above framework associates to the initial problem (5.1) (5.2) (5.7) an element ϕ of X which is the Lagrange multiplier for the constrained optimization problem (5.7).

The necessary condition on b which guarantees the existence of ϕ, the so-called inf-sup condition, was introduced independently by Babuska [2] and Brezzi [5] and reads

$$(5.8)\qquad \begin{cases} \text{There exists} \quad \beta > 0 \quad \text{such that} \\ \\ \inf_{\psi\in Y} \sup_{v\in X} \dfrac{b(v,\psi)}{\|v\|_X \|\psi\|_Y} \geq \beta . \end{cases}$$

Equivalently (5.8) means that the linear operator B from X into Y' defined by

$$(5.9)\qquad \langle Bv,\psi\rangle = b(v,\psi), \quad \text{for all } v\in x, \text{ for all } \psi\in Y'$$

is an isomorphism from the orthogonal of V in X onto Y' or that the adjoint B' of B which maps X' into Y is an isomorphism from X onto the polar set V^o of V

$$V^o = \{\theta\in X', \ \langle\theta,v\rangle = 0, \text{ for all } v\in V\}.$$

The reader is referred for more details to the article of Brezzi in this volume. Note that the form (5.6) of the problem can be studied independently of the corresponding Lagrangian and variational problems and is suitable for several types of generalizations:

- Given a linear continuous form χ on Y, we can replace the second equation (5.6) by

$$b(u,\psi) = \langle \chi, \psi \rangle, \quad \text{for all } \psi \in Y$$

- More important, the form a may be nonlinear with respect to its first argument u, and this corresponds to considering nonlinear partial differential equations, in particular the Navier-Stokes equations (see below) or monotone operators (see [24]).

See also in [11] a different point of view for duality which includes (5.6) as a particular case.

Remark 5.1. - Let us mention also here the penalization of (5.6) which leads to consideration of the following problems

$$(5.10) \qquad \begin{cases} \text{To find } u^\varepsilon \in X, \ \phi^\varepsilon \in Y \text{ such that} \\ a(u^\varepsilon, v) + b(v, \phi^\varepsilon) = \langle \ell, v \rangle, \quad \forall\, v \in X \\ -\varepsilon c(\rho^\varepsilon, \psi) + b(u^\varepsilon, \psi) = 0, \quad \text{for all } \psi \in Y \end{cases}$$

where c is a bilinear continuous coercive symmetric form on Y and $\varepsilon > 0$ is a fixed positive parameter which is intended to tend to 0. A solution $u^\varepsilon, \phi^\varepsilon$ of (5.10) exists for every $\varepsilon > 0$, and $u^\varepsilon, \phi^\varepsilon$ converges to the solution u, ϕ of (5.6) when $\varepsilon \to 0$; see M. Bercovier [3].

5.b. Examples

Stokes Equations

The Stokes equations provide one of the most typical examples where the framework (5.6) is suitable. Stokes problem is the problem (1.7) - (1.10) when $U = 0$ and the nonlinear term $(u\cdot\nabla)u$ is dropped. In this case ($\Omega \subset \mathbb{R}^n$, $n = 3$ or more generally $n \neq 3$):

$$X = H_0^1(\Omega)^n = \{v \in L^2(\Omega)^n,\ \frac{\partial v}{\partial x_i} \in L^2(\Omega)^n,\ \forall i,\ v = 0 \text{ on } \Gamma\}$$

$$V = \{v \in X,\ \text{div } v = 0\}$$

$$Y = \{\phi \in L^2(\Omega),\ \int_\Omega \phi(x)\, dx = 0\}$$

$$a(u,v) = \nu \sum_{i,j=1}^{n} \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}\, dx$$

$$b(v,\phi) = -\int_\Omega (\text{div } v)\, \phi\, dx.$$

It can be shown that (5.6) is equivalent to the Stokes problem

$$\text{(5.10)} \qquad \begin{cases} -\nu\Delta u + \text{grad } p = f & \text{in } \Omega \\ \text{div } u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma. \end{cases}$$

The operator div is a surjection from V onto Y (see for instance R. Temam [29]), and it follows immediately that the Babuska-Brezzi condition (5.8) is satisfied.

As indicated above, we can set the Navier-Stokes equations in the framework (5.6) with a replaced by a nonlinear form

$$a(u,v) = a_0(u,v) + a_1(u,v)$$

$$a_0(u,v) = \nu \sum_{i,j=1}^{n} \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx$$

$$a_1(u,v) = \sum_{i,j=1}^{n} \int_\Omega u_i \frac{\partial u_j}{\partial x_i} v_j dx.$$

In that case cf. [6].

Dirichlet Problem

The framework (5.6) applies to (3.5) and provides the dual of this problem (see Ekeland-Temam [11]). For simplicity we restrict ourselves to the case where $\Gamma_0 = \Gamma$, $\Gamma_1 = \emptyset$, i.e., we consider the Dirichlet problem in Ω. We set

$$X = H_0^1(\Omega) \times L^2(\Omega)^n, \quad Y = L^2(\Omega)^n$$

$$V = \{u = \{u_0,u_1\} \in X, \ u_1 = \text{grad } u_0\}$$

and for $u = \{u_0,u_1\}$, $v = \{v_0,v_1\} \in X$ and $\psi \in Y$:

$$a(u,v) = \int_\Omega u_0,v_0 dx + \int_\Omega u_1 \cdot v_1 dx$$

$$\langle \ell, v \rangle = \int_\Omega f v_0 dx$$

$$b(v,\psi) = \int_\Omega (v_1 - \text{grad } v_0)\cdot\psi dx.$$

We identify (5.6) with (3.5). Condition (5.8) is trivially satisfied. The dual (5.5) reads

$$(5.11)\qquad \begin{cases} \text{To maximize} \quad -\frac{1}{2}\int_\Omega |\psi|^2 dx, \\ \text{for } \psi\in L^2(\Omega)^n, \ \text{div } \psi + f = 0 \ \text{ in } \Omega. \end{cases}$$

Biharmonic Problem

We consider the problem

$$(5.12)\qquad \Delta^2 u = f \ \text{ in } \Omega,$$

$$(5.13)\qquad u = 0, \ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \Gamma.$$

It is set in the form (5.6) with

$$V = H_0^2(\Omega) = \{v\in L^2(\Omega), \ \frac{\partial v}{\partial x_i}, \ \frac{\partial^2 v}{\partial x_i \partial x_j}\in L^2(\Omega), \text{ for all } i,j, \text{ and } v = 0, \ \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma\}$$

$$a(u,v) = \int_\Omega \Delta u \, \Delta v \, dx$$

$$\langle \ell, v\rangle = \int_\Omega fv dx.$$

Then we set it in the form (5.6) with

$$X = H_0^2(\Omega) \times L^2(\Omega), \quad Y = L^2(\Omega)$$

$$V = \{u = \{u_0, u_1\} \in X, \ u_1 = \Delta u_0\}$$

and for $u = \{u_0, u_1\}$, $v = \{v_0, v_1\} \in X$ and $\psi \in Y$

$$a(u,v) = \int_\Omega u_1 v_1 dx, \qquad \langle \ell, v \rangle = \int_\Omega f v_0 dx$$

$$b(v,\psi) = \int_\Omega (\Delta v_0 - v_1) \cdot \psi dx.$$

We identify (5.6) with (3.5). Condition (5.8) is trivially satisfied. The dual (5.5) reads

$$(5.14) \qquad \begin{cases} \text{To maximize} \quad -\frac{1}{2} \int_\Omega |\psi|^2 dx \\ \text{for } \psi \in L^2(\Omega), \ \Delta\psi = f \text{ in } \Omega. \end{cases}$$

Problems involving the biharmonic appear in elasticity and in fluid mechanics for the treatment of the Stokes problem by utilization of a stream function.

5.c. Mixed and Hybrid Elements

Once we have reduced Problem (5.1) to (5.6), we are naturally led to approximate this last problem, i.e.,

- To find X_h, Y_h, a_h, b_h, c_h which approximate X, Y, a, b, c.
- Solve for each h a discrete problem similar to (5.6):

$$
(5.15)\qquad \begin{cases} \text{To find } u_h \in X_h,\ \phi_h \in Y_h \text{ such that} \\ a_h(u_h, v_h) + b_h(v_h, \phi_h) = \langle \ell_h, v_h \rangle \quad \text{for all } v_h \in X_h \\ b_h(u_h, \psi_h) = 0, \quad \text{for all } \psi_h \in Y. \end{cases}
$$

Finite element methods appear naturally in the construction of the spaces X_h and Y_h. We have more flexibility than in an ordinary finite element method since we can combine various finite elements for X_h and for Y_h. The major difficulty arises in the verification of the condition (5.8) which leads sometimes to delicate algebraic questions. A thorough investigation of the inf sup condition for various finite elements related to the Navier-Stokes equations can be found in J. T. Oden and O. P. Jacquotte [21]. In some cases, the number β in (5.8) corresponding to the discrete case depends on h and tends to 0 as $h \to 0$. In other cases, the inf sup condition does not hold in the discrete case and we can make it true by reducing the space Y_h in order to suppress the kernel of the discrete analogue B'_h of B'; in practice this amounts to a filtering procedure. The most famous example is the classical checkerboard instability for a Stokes problem corresponding to the $Q_1 - P_0$ element: X_h is a Q_1 approximation of $H_0^1(\Omega)^2 (n = 2)$ and Y_h is a P_0 approximation of $L^2(\Omega)$; the filtering procedure is standard in this case; see also the analysis of the inf sup condition in Boland-Nicolaides [4] who show with a counter example that the best value of β in this (discrete) case is of the form $c\,h$.

We will not develop further this question here since it is the object of [6] and other articles in this volume.

SPECIFIC APPLICATIONS

Finite element methods have been the object of many applications in mechanics and physics. We describe now some specific applications.

6. Navier-Stokes Equations

The notations being the same as in Section 1 we consider the time dependent Navier-Stokes equations for viscous incompressible flow in a domain Ω

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad \text{in } \Omega \times (0,T) \tag{6.1}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega \times (0,T) \tag{6.2}$$

$$u = U \quad \text{on } \Gamma \times (0,T). \tag{6.3}$$

The unknowns are the velocity vector $u = u(x,t)$ and the pressure $p = p(x,t)$, the volumic forces f and the boundary velocity U (which may both depend on time) are given.

We recall that from the mathematical point of view the initial value problem for the Navier-Stokes equations, i.e., (6.1) - (6.3) supplemented by an initial condition

$$u(x,0) = u_0(x), \ x\in\Omega, \ u_0 \text{ given}, \tag{6.4}$$

is well set in space dimension 2 ($\Omega \subset \mathbb{R}^2$). However, we do not know yet if the same result is true in space dimension 3 ($\Omega \subset \mathbb{R}^3$), i.e., we do not know if

the curl vector remains bounded or may become infinite even if the data are regular; see for instance R. Temam [29].

The interval of time that we consider may be finite or infinite. Finite intervals of time occur naturally in the study of transient phenomena, while "infinite" intervals of time appear in the study of permanent regimes. For instance if f and U are independent of time, then in some cases the solution u, p of (6.1) - (6.4) converges as $t \to \infty$ to a stationary solution, i.e., a solution of (1.7) - (1.9) . A sufficient condition for this to occur is that the Reynolds number R_e is sufficiently small

$$R_e = \frac{U_* L_*}{\nu}$$

where U_* is a typical velocity of the flow and L_* a typical length of Ω[2]. If R_e is large, the convergence to a stationary solution is not guaranteed anymore. Based on experimental observations relative to turbulence, we actually expect that $u(\cdot,t)$, $p(\cdot,t)$ do not converge anymore to time independent solutions even if the data f,U are independent of time. From the numerical point of view this will be the source of new difficulties which have not yet been explored and will not be addressed here (R. Temam [30]). Actually the computing power that is presently available leaves us at the threshold of the occurrence of nonstationary phenomena at least in space dimension two.

---

[2]Typical velocities are provided by ν and f in the form $L_*^{\alpha_1}\nu^{\alpha_2}$ norm(U), $L_*^{\beta_1}\nu^{\beta_2}$ norm(f), where appropriate norms are considered and the α_i, β_i are such that the corresponding expressions have the dimension of a velocity. The sum of these two velocities is an appropriate definition of U_*.

Up to now most of the numerical computations on the full Navier-Stokes equations dealt with stationary phenomena (and transient phenomena). At this level a major difficulty for the numerical solution of (6.1) - (6.4) (or (1.7) - (1.9)) is the handling of the free divergence conditions which introduce complicated algebraic conditions in the discrete problem if it is not treated correctly. A certain number of methods related to or independent from the finite elements have been proposed to overcome the difficulties associated with the condition div $u = 0$.

a) Utilization of the Penalty Method

The penalty method which is due to R. Courant [10] in the context of constrained optimization was applied to the Navier-Stokes equations in R. Temam [25], [26]. The idea which stems from the variational form of the Stokes problem (see Section 5) is to treat the condition div $u = 0$ as a constraint and to "penalize" it, i.e., to replace (6.1)(6.2) by

$$\frac{\partial u_\varepsilon}{\partial t} - \nu\Delta u_\varepsilon + (u_\varepsilon \cdot \nabla)u_\varepsilon - \frac{1}{\varepsilon}\nabla(\nabla \cdot u_\varepsilon) = f \quad \text{in} \quad \Omega x \quad (0,T) \tag{6.5}$$

where $\varepsilon > 0$ is a small parameter which is intended to tend to 0. It can be proved [26] that the solution of (6.3) - (6.5) converges to that of (6.1) - (6.4) when $\varepsilon \to 0$. A full asymptotic expansion of u^ε, p^ε in terms of ε can even be obtained in the simpler case of Stokes flows [29].

The penalty method has been applied in several ways by many authors to the finite element approximations of the Navier-Stokes equations, in particular with the mixed finite elements (see Remark 5.1).

b) Utilization of Algorithms: Artificial Time Dependence

This method applies to the solution of the stationary problem (1.7) - (1.9) or to the solution of the stationary problems arising from time discretization of (6.1) - (6.4). In these cases, an artificial evolution problem is introduced whose stationary solution is also the solution of the stationary Navier-Stokes equation.

For instance, one can consider the artificial evolution problem

$$\text{(6.6)} \qquad \begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \text{grad } p = f \\ \dfrac{\partial p}{\partial t} + \alpha \text{ div } u = 0 \end{cases}$$

($\alpha > 0$) or the equations of slightly compressible fluids

$$\text{(6.7)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \text{grad } p = f \\ \dfrac{\partial p}{\partial t} + \alpha \text{ div } u = 0. \end{cases}$$

In both cases the condition div u = 0 is not imposed at all times and follows simply from the properties of (6.6) and (6.7) for large t. Consequently, ordinary finite elements are used, i.e., finite elements not containing the condition div u = 0.

c) Utilization of the Projection Method

This method introduced in A. J. Chorin [8] and R. Temam [27] is connected to the fractional step method. It consists in solving the time evolution of

(6.1) without (6.2) and then, more or less frequently, imposing (6.2) by projecting the velocity obtained on the free divergence vector fields.

The time discretization (time mesh = Δt) when $U = 0$ is given by

$$(6.8)\qquad \begin{cases} \dfrac{\tilde{u}^m - u^{m-1}}{\Delta t} - \nu\Delta\tilde{u}^m + (\tilde{u}^m\cdot\nabla)\tilde{u}^m = f^m \quad \text{in} \quad \Omega \\ \tilde{u}^m = U^m \quad \text{on} \quad \Gamma \end{cases}$$

and u^m = Proj. of $\tilde{u}^m$ which amounts to saying that

$$(6.9)\qquad \begin{cases} u^m = \tilde{u}^m - \text{grad } q^m \quad \text{in} \quad \Omega \\ \text{div } u^m = 0 \quad \text{in} \quad \Omega \\ u^m\cdot\underline{\nu} \ (= \text{normal component of } u^m \text{ on } \Gamma) = 0 \quad \text{on} \quad \Gamma. \end{cases}$$

Alternatively setting $q^m = \Delta t\, \pi^m$ we can rewrite (6.9) as

$$(6.10)\qquad \begin{cases} \dfrac{u^m - \tilde{u}^m}{\Delta t} + \text{grad } \pi^m = 0 \quad \text{in} \quad \Omega \\ \text{div } u^m = 0 \quad \text{in} \quad \Omega \\ u^m\cdot\underline{\nu} = 0 \quad \text{on} \quad \Gamma, \end{cases}$$

and p^m provides an approximation for the pressure. It is, however, a poor approximation since it satisfies the following nonphysical boundary condition

that we infer from (6.10):

$$\frac{\partial \pi^m}{\partial \underline{\nu}} = 0 \quad \text{on} \quad \Gamma. \tag{6.11}$$

In order to determine u^m it is necessary to actually compute q^m, π^m, or at least their gradient; π^m is a solution of the Neuman problem which consists of (6.11) and

$$\Delta \pi^m = \frac{1}{\Delta t} \operatorname{div} \tilde{u}^m. \tag{6.12}$$

It seems better, for a more accurate determination of the pressure which avoids the undesirable boundary layer resulting from (6.11), to consider q^m, π^m as auxiliary functions and to compute the approximation p^m of the pressure by using the boundary value problem

$$p = \psi(u) \qquad p^m = \psi(u^m) \tag{6.13}$$

that we deduce directly from the Navier-Stokes equations; cf. [29]. At this point one can either use a Dirichlet or a Neuman boundary condition for p [15].

Many other forms of (6.8)(6.9) can be also considered: one can split the operators differently, leaving for example some viscosity in (6.9), one can use an explicit scheme in (6.8), or one can solve for the evolution (6.8) for several steps and perform the projection (6.9) periodically only.

In all cases when the projection method is used we need a space of free divergence vector functions so that the projection (6.9) can be performed in a satisfactory manner.

d) Utilization of free divergence finite element spaces

The simplest element, the piecewise linear (P_1) function on triangles, cannot be used since the condition div u = 0 imposed on each triangle leads to too many algebraic relations and the spaces of discrete divergence free P_1 vector functions may be reduced to the function 0. One can either impose the condition div u = 0 "less often," or go to more complicated elements such as the nonconforming P_1 element (F. Thomasset) or P_2, Q_1, Q_2, ..., elements.

7. Fluid Structure Interactions

In many industrial fields of interest, including the space industry, engineers are confronted with fluids (water, oil, kerosene, gases, ...) interacting with structures (tanks, containers, obstacles, ...) along a more or less extended area.

In some cases deformations of the structure may be fairly important and even affect the motion of the fluid; engineers have then to solve problems including a coupling between fluid displacements and elastic deformations of the structure.

We describe here the interaction between a free surface fluid and the structure, assumed to be elastic, which contains it, in an external force field. We follow J. Mathieu [13] and J. Mathieu, et al. [20], who computed the transient simulation of such a process with the BACCHUS Code.

7.a. The Arbitrary Lagrange-Euler (ALE) Description

We consider a moving domain $\Omega^F(t)$ deforming with velocity $w = w(x,t)$, and filled with an incompressible fluid of (constant) density

ρ^F which obeys the Navier-Stokes equations. We consider also a second domain $\Omega^S(t)$ made of elastic material and limiting Ω^F. The fluid is limited by a free surface S and the contact surface $\pi = \pi(t)$ with Ω^S.

The equations are

$$(7.1)\quad \begin{cases} \rho^F \left\{ \frac{\delta}{\delta t} v^F + ((v^F - w)\nabla)v^F \right\} = \operatorname{div} \sigma^F + \rho^F f^F \quad \text{in } \Omega^F \\ \operatorname{div} v^F = 0 \\ \rho^S \frac{\partial v^S}{\partial t} = \operatorname{div} \sigma^S + \rho^S f^S \quad \text{in } \Omega^S \\ \frac{d}{dt} (\rho^S |J|) = 0 \end{cases}$$

where

$\frac{\delta \rho}{\delta t} = \frac{\partial \rho}{\partial t} + (w \cdot \nabla)\rho$ is the convection derivative associated with the vector field w

v^i = the velocity field in Ω^i

σ^i = the stress tensor in Ω^i

ρ^i = density in Ω^i (constant in Ω^F)

f = external forces

J = Jacobian of the mapping $\Omega^S(0) \to \Omega^S(t)$.

The ALE description is determined by the actual velocity field w which is defined as follows:

$w = 0$ in an internal eulerian region

$w = v^F$ on the free surface $S(t)$

$w = v^S$ on the wet part of the wall $\pi(t)$.

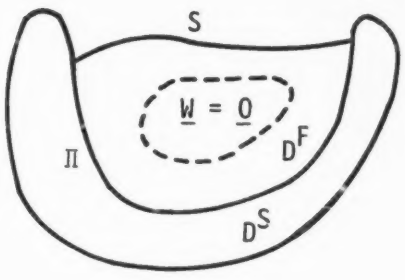


Figure 7.1

The constitutive laws are

$$\sigma^F = \nu[\underset{\sim}{\nabla} v^F + (\underset{\sim}{\nabla} v^F)^t] + p\underset{\sim}{I}$$

for the fluid, ν = the kinematic viscosity, p = the hydrostatic pressure, and

$$\sigma^S(t) = \frac{\rho^S(t)}{\rho^S(o)} F(t)\, \underset{\sim}{\tau}(t)\, F(t)^t$$

for the solid where $\underset{\sim}{\tau}$ is the standard second Piola Kirchhoff tensor, F(t) is the gradient of the mapping $\Omega^S(0) \to \Omega^S(t)$, and $\underset{\sim}{\tau}(0) = \sigma^S(0) = 0$.

The boundary conditions are as follows:

- On the structure, displacements are given on some part $\Gamma_u(t)$ of $\partial\Omega^S(t) \backslash \pi(t)$ and normal stresses are given along the remaining part of $\partial\Omega^S(t) \backslash \pi(t)$.
- On the wet part of the wall, $\pi(t)$, we have no cavitation,

$$v^F \cdot \underline{\nu} = v^S \cdot \underline{\nu}$$

$\underline{\nu}$ the normal on π, and we have a partial slip condition of the fluid along the wall

$$\sigma^F_{tg} = -\beta(v^F_{tg} - v^S_{tg})$$

and finally the normal stresses are continuous

$$(\sigma^F - \sigma^S) \cdot \underline{\nu} = 0.$$

The discretization of the problem is made with a finite element discretization in space, providing an easy handling of the complex geometric configurations which are caused by large displacements. The elements are of degree one for the velocities (= P_1 for triangles, Q_1 for rectangles, Q_1 transported Q_1 for quadrilaterals), and piecewise constants for the hydrostatic pressure.

A one-step explicit finite difference scheme is used in time. This scheme is subject to the usual stability conditions limiting the time step.

The criteria taken into account are

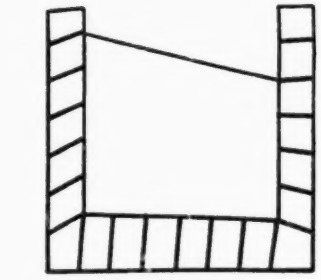
- free surface wave and viscous wave stability in the fluid
- acoustic wave stability in the structure.

Because of the more drastic limitation due to the stability criterion in the structure, a subcycling procedure is used for the structure calculation.

A mesh adaptation procedure is necessary. At each fluid-calculation step, the free surface has to be repositioned. The displacements of the free surface and of the interface between the fluid and the structure induce a modification of the mesh in the mixed Lagrange-Euler region and possibly a modification of the Euler region (when its boundary intersects the free surface or the wall) or a degeneracy of some element. Thus a rezoning is automatically performed during the calculation. The rezoning is also necessary on $\pi(t)$ since any node belonging to $\partial\Omega^F(t) \cap \partial\Omega^S(t)$ should be at the same time a vertex of some element of $\Omega^S(t)$ and of some element of $\Omega^F(t)$.

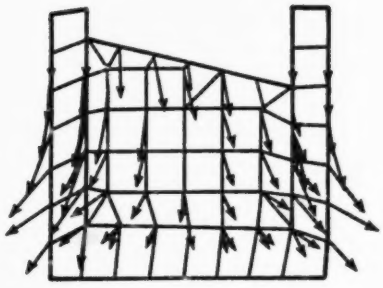
A sample two-dimensional calculation is shown on Figure 7.2:

-47-



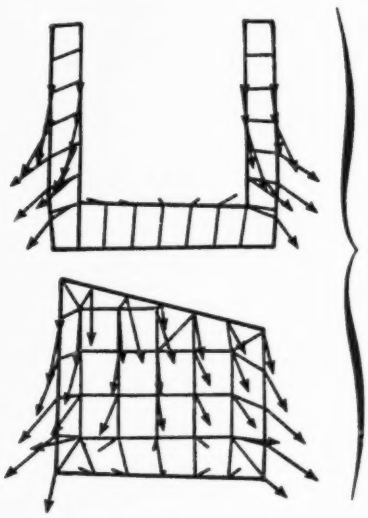
t = 0 — D^S and S are given.

The internal mesh will be automatically designed.

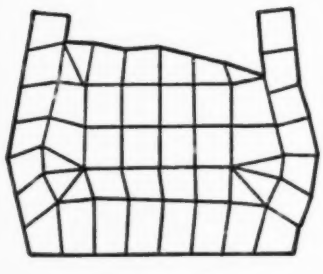


t = 0,2 — The bottom of the tank is in a compressive phasis.

Notice the behavior of the velocity fields on π: double-valued with normal component continuity.



t = 0,3 — The bottom of the tank has entered an expansive phasis.



t = 0,5 — The tank has reached large enough deformations.

The free-surface tends to stabilize towards the horizontal. Notice the modification of the mesh in D^F.

Figure 7.2

8. Three Dimensional Euler Equations

The finite element method has been applied to the computation of the solution of the Euler equations. We describe here the latest results in C. H. Bruneau, et al. [7], which concern the computation of steady vortex flows past a flat plate at high angle of attack.

8.1. Description of the Problem

The purpose of the computations in [7] is to investigate the developments of vortices at the tip of the plate and their propagation after the trailing edge. In the computations, the plate has no thickness and we expect a strong vortex structure to develop at the tip of the plate; it is not possible to use potential equations and the full Euler equations are necessary. For the computations, the plate is imbedded in a 3-D rectangular domain as shown on Figure 8.1; the aspect ratio is 0.5 and the angle of incidence is $\alpha = 15^{o}$ or $\alpha = 30^{o}$. Figure 8.1 shows only half of the plate since symmetry with respect to the y variable is assumed.

The incoming flow is given by $\vec{q}_{\infty} = (u_{\infty}, v_{\infty}, w_{\infty}) = (q_{\alpha} \cos\alpha, o, q_{\alpha} \sin\alpha)$, α as above, and M_{α} the Mach number at infinity, $M_{\infty}^2 = \frac{q_{\infty}^2}{a_{\infty}^2}$, $a_{\infty}^2 = (\gamma - 1)(H - \frac{q_{\infty}^2}{2})$. In these computations, the flow is subsonic, $M_{\infty} = 0.7$.

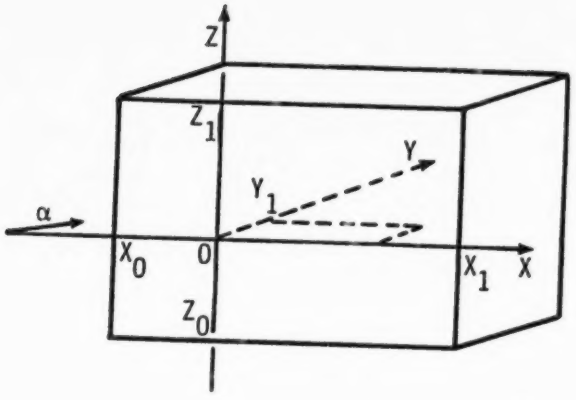


Figure 8.1

Euler Equations

They are written in conservative form

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \qquad \text{Conservation of mass}$$

$$\frac{\partial \rho u^2 + p}{\partial x} + \frac{\partial \rho uv}{\partial y} + \frac{\partial \rho uw}{\partial z} = 0$$

(8.1)

$$\frac{\partial \rho uv}{\partial x} + \frac{\partial \rho v^2 + p}{\partial y} + \frac{\partial \rho vw}{\partial z} = 0 \qquad \text{Conservation of momentum}$$

$$\frac{\partial \rho uw}{\partial x} + \frac{\partial \rho vw}{\partial y} + \frac{\partial \rho w^2 + p}{\partial z} = 0$$

(8.2) $$\frac{\gamma p}{(\gamma - 1)\rho} + \frac{q^2}{2} = H \qquad \text{Bernoulli's equation}$$

where ρ, $q^{-s} = (u,v,w)$, p represents respectively the density, the velocity vector, and the pressure; H is the total and γ the ratio of specific heats.

Boundary Conditions

The boundary conditions on the plate are easy; the tangency condition on the plate (plane $z = 0$) reads $w = 0$, and the symmetry condition is $v = 0$ on $y = 0$. On the contrary, the boundary conditions at the limits of the domain are not easy, especially downstream where it should allow the vortex to go through the exit plane.

The flow variables have been set to the freestream values at the incoming boundaries ($x = x_0$, $z = z_0$), and no condition was imposed at the exit boundaries ($x = x_1$, $z = z_1$) so that the values there are computed from the variational formulation. Due to the utilization of a least square method, this amounts simply to requiring that the first order equations be satisfied at those boundaries (see [7] for a discussion of the boundary conditions). At $y = y_1$ the far field conditions $v = 0$ is used.

8.2 The Numerical Method and the Numerical Results

The equations are solved iteratively as follows:

- A fixed point algorithm is based on equation (8.2), computing the density when u, v, w, p are known.
- The nonlinear system (8.1) provides u, v, w, p, using the value of ρ from the previous step of the iteration. This system is linearized by Newton's method with linearized variables $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $\tilde{p}$.
- The system for $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $\tilde{p}$ is discretized by an appropriate Q_1 finite element method (discretization of the conservative variables ρu, ρv, ρw, ...) and the system is solved by a least square method.

Figures 8.2 and 8.3 show a sample of the discretization grid. Finally, Figures 8.4 to 8.6 show the cross flow velocities at 70%, 90%, and 110% of the plate

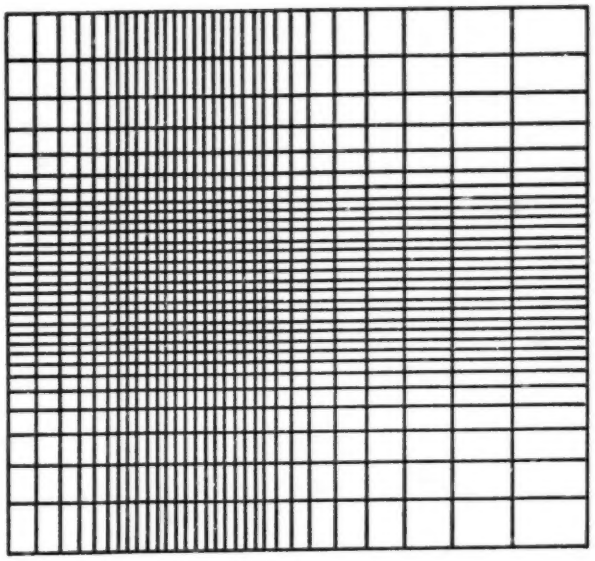


Figure 8.2

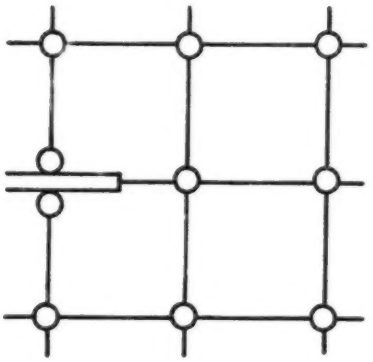


Figure 8.3

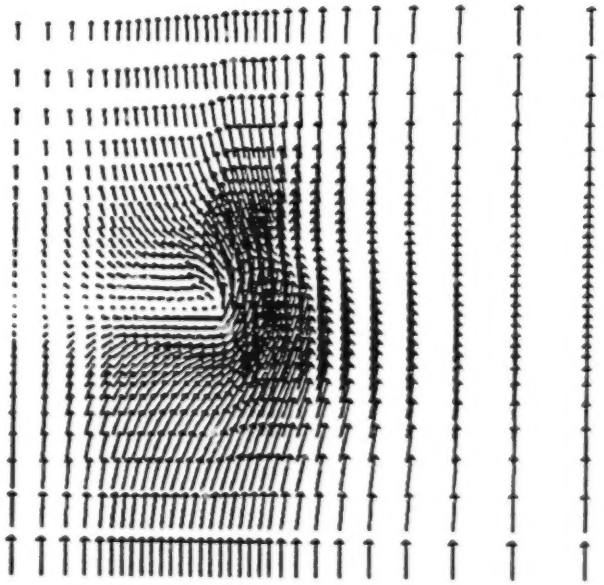


Figure 8.4

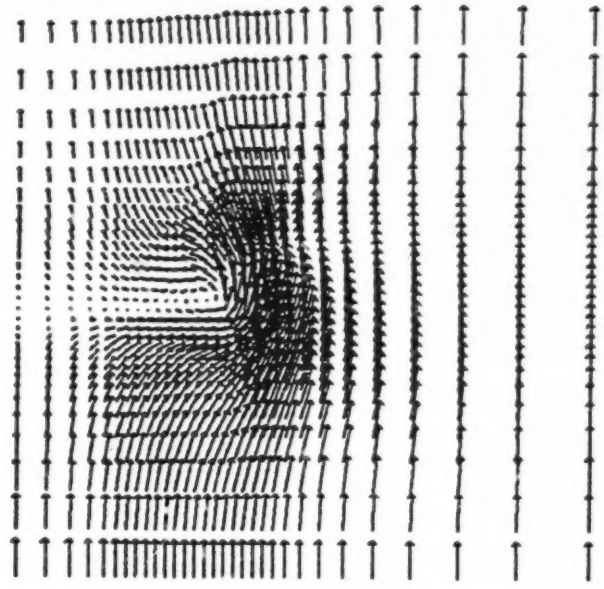


Figure 8.5

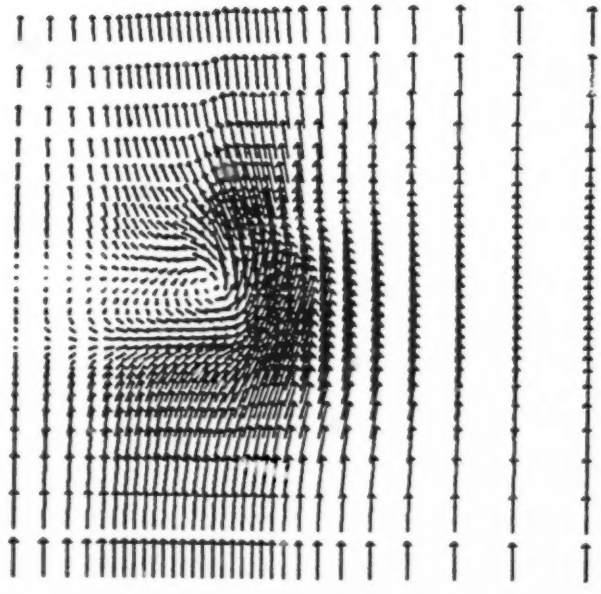


Figure 8.6

FUTURE DEVELOPMENTS

The advantages of FEM are well known. In particular

- accurate and automating fitting of complicated geometries.
- lowering (dividing by 2) the order of the differential operators that appear, due to the utilization of weak formulations.
- avoiding the discretization of the boundary conditions of the Neuman type which disappear in the weak formulations.

The inconveniences are

- the need of a triangulation program for domains which has to be written once for all but is a discouraging preliminary step for the nonexpert.
- the computational costs which by no way can compete with the achievements of multigrid and spectral methods.

However, the multigrid and spectral methods attain their optimal efficiency for rectangular domains. It is then conceivable that a combination of FEM and multigrid and spectral methods in relation with domain decomposition and parallel computation can prove to be efficient.

9. Domain Decomposition: Remarks on Future Developments

The domain decomposition which lies at the foundation of the FEM can appear to be, in a different form, directly related to future developments in the method. Besides the geometrical considerations, there are many other good reasons for decomposing the solution of a boundary value problem in a large domain into the solution of similar problems in subdomains. Let us mention some of them:

a) Adaptive meshes

Adaptive meshes may be suitable for geometrical or analytical reasons. For instance, the fitting of a curved domain with elements of different nature which are more appropriate in various parts of the domain. The refinement of the mesh in regions where the solutions are singular or nearly so (boundary layers, shocks, front flames, etc.) allow for extra computational effort to be concentrated in such regions. Thus finite elements of different types and sizes may be used in various parts of a domain, and a different treatment of the different regions can be useful.

b) Physical Motivations

Physical phenomena of a different nature may occur in different regions, and these regions should not be treated in a similar manner. For instance in aeronautics

- a turbulence model is necessary near the airfoil
- the Navier-Stokes equations with viscous effects and without turbulence are necessary at a certain distance but not too far
- the Euler equations (i.e., no viscous effect) are sufficient far from the airfoil.

Similar situations occur in combustion or in solid mechanics and lead naturally to the decomposition of the whole domain into smaller ones.

c) Parallel Computation

The technology of computers will apparently move more and more toward parallel computation. The decomposition of a domain into subdomains on which the computations are made simultaneously is appropriate for parallel computa-

tions. The difficulty is then to periodically synthesize the information from all the subdomains to ensure a correct interaction of the subdomains.

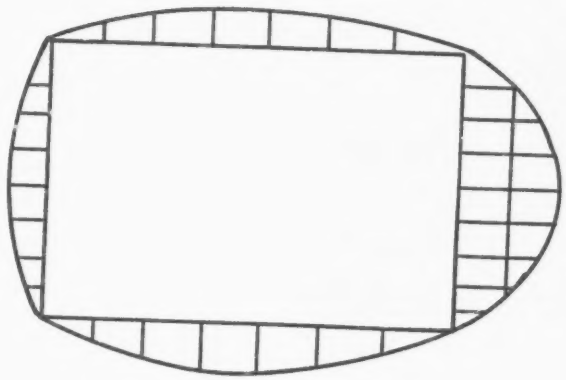


Figure 9.1

At the present time work is being done on domain decomposition. If this technique is satisfactorily mastered, then one can consider, as has already been done (see for instance [13][23][19]), combining the advantages of FEM and spectral and multigrid methods by using domain decomposition as suggested in Figure 9.1. This has already been done, but the goal is to get as close as possible to the performance of the fast methods.

CONCLUDING REMARK

The FEM cannot pretend to be the best method in all situations, but it has proved to be an efficient and performant method in many cases and can hope to be the object of future interesting developments. R. Feyman says in his book [12] that he was able to solve some problems that other people could not solve just because he had some tools in his box that others did not have. It would be too bad not to have the FEM tool in his box of numerical methods.

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Final Report

16. Abstract

The finite element methods (FEM) have proved to be a powerful technique for the solution of boundary value problems associated with partial differential equations of either elliptic, parabolic, or hyperbolic type. They also have a good potential for utilization on parallel computers particularly in relation to the concept of domain decomposition.

This report is intended as an introduction to the FEM for the nonspecialist. It contains a survey which is totally nonexhaustive, and it also contains as an illustration, a report on some new results concerning two specific applications, namely a free boundary fluid-structure interaction problem and the Euler equations for inviscid flows.

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